Remarks concerning the

ENERGETICS OF A GAUSSIAN WAVEPACKET

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Introduction. I have had occasion to write at extravagant length about the quantum mechanical evolution of initially Gaussian wavepackets.[1] The recent appearance of a paper by Katsunori Mita[2] has served, however, to remind me of an aspect to the topic which I have previously neglected to consider. It is my intention in this short essay to remedy that omission, and to frame an opinion concerning the value of the ideas put forward by Mita.

We look to the quantum mechanics of a mass point m that moves freely in one dimension; *i.e.*, to the system

$$\mathsf{H} = \tfrac{1}{2m}\,\mathsf{p}^2$$

Because momentum is a constant of the motion ($[\mathsf{H}, \mathsf{p}] = \mathbf{0}$) it is simplest to work in the momentum representation, where the Schrödinger equation reads

$$\tfrac{1}{2m}\,p^2 \varphi(p,t) = i\hbar \partial_t \varphi(p,t)$$

and leads immediately to

$$\varphi(p,t) = \varphi(p) \cdot e^{-\frac{i}{\hbar}(p^2/2m)t}$$

where $\varphi(p) = \varphi(p,0)$ is subject only to the normalization condition

$$\int |\varphi(p)|^2 dp = 1$$

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[1] See "Gaussian wavepackets" (1998), especially pages 2–12. I will be at pains to adhere here to the notations adopted there.

[2] "Dispersive properties of probability densities in quantum mechanics," AJP **71**, 894 (2003).

I restrict my attention here to the (p_0, λ)-parameterized class of cases

$$\varphi(p; p_0, \lambda) \equiv \left[\tfrac{1}{\lambda\sqrt{2\pi}}\right]^{\frac{1}{2}} \exp\left\{-\tfrac{1}{4}\left[\tfrac{p-p_0}{\lambda}\right]^2\right\}$$

Passing by Fourier transformation to the **x**-representation, we find

$$\psi(x, t) = \tfrac{1}{\sqrt{h}} \int e^{+\frac{i}{\hbar}px} \left\{\left[\tfrac{1}{\lambda\sqrt{2\pi}}\right]^{\frac{1}{2}} \exp\left\{-\tfrac{1}{4}\left[\tfrac{p-p_0}{\lambda}\right]^2\right\}\right\} \cdot e^{-\frac{i}{\hbar}(p^2/2m)t}\, dp$$

$$\equiv \left[\tfrac{1}{\sigma[1+i(t/\tau)]\sqrt{2\pi}}\right]^{\frac{1}{2}} \exp\left\{-\tfrac{x^2}{4\sigma^2[1+i(t/\tau)]} + \tfrac{i}{\hbar}\tfrac{p_0 x - (p_0^2/2m)t}{1+i(t/\tau)}\right\}$$

$$= \left[\text{etc.}\right]^{\frac{1}{2}} \exp\left\{\tfrac{1}{4\sigma^2(t)}\left[-(x-vt)^2 + i\tfrac{t}{\tau}(x^2 - v^2\tau^2)\right]\right\} \cdot \exp\left\{\tfrac{i}{\hbar}\tfrac{mvx}{1+(t/\tau)^2}\right\}$$

where $v \equiv p_0/m$ is a "velocity" of arbitrary value, where the parameters τ, λ and σ—which are dimensionally a "time," a "momentum" and a "length"—stand in the relationships

$$\tau \equiv \hbar m/2\lambda^2 \equiv 2m\sigma^2/\hbar$$

and where

$$\sigma(t) \equiv \sigma\sqrt{1 + (t/\tau)^2}$$
$$\geqslant \sigma$$

It is evident that

$$|\psi(x, t)|^2 = \tfrac{1}{\sigma(t)\sqrt{2\pi}} \exp\left\{-\tfrac{1}{2}\left[\tfrac{x-vt}{\sigma(t)}\right]^2\right\} \tag{1.1}$$

and

$$|\varphi(p, t)|^2 = \tfrac{1}{\lambda\sqrt{2\pi}} \exp\left\{-\tfrac{1}{2}\left[\tfrac{p-mv}{\lambda}\right]^2\right\} \quad : \quad \text{all } t \tag{1.2}$$

both describe normal distributions, and that

$$\langle \mathbf{x}\rangle = vt \qquad\qquad\qquad \langle \mathbf{p}\rangle = mv$$
$$\Delta x \equiv \sqrt{\langle \mathbf{x}^2\rangle - \langle \mathbf{x}\rangle^2} = \sigma(t) \qquad \Delta p \equiv \sqrt{\langle \mathbf{p}^2\rangle - \langle \mathbf{p}\rangle^2} = \lambda$$

At $t = 0$ we have the "minimal dispersion relation"

$$\Delta x \cdot \Delta p = \sigma\lambda = \tfrac{1}{2}\hbar$$

1. Expected results of energy measurements. From

$$\bar{E} \equiv \langle \mathbf{H}\rangle = \tfrac{1}{2m}\langle \mathbf{p}^2\rangle = \tfrac{1}{2m}\left\{\langle \mathbf{p}\rangle^2 + (\Delta p)^2\right\}$$

it is immediate that for a free Gaussian wavepacket

$$\left.\begin{aligned} \bar{E} &= \tfrac{1}{2}mv^2 + \lambda^2/2m \\ &= \tfrac{1}{2}mv^2 + \hbar^2/8m\sigma^2 \\ &= \tfrac{1}{2}mv^2 + \hbar/4\tau \end{aligned}\right\} \tag{2}$$

This result is susceptible to several modes of intuitive interpretation. We might,

for example, write

$$\bar{E} = \text{classical drift energy} + \text{quantum correction}$$

where both terms are constant, and the latter remains present even in the absence of drift ($v = 0$). The "quantum correction" $\hbar^2/8m\sigma^2$ can be understood physically as the *energy that was <u>invested in the assembly</u> of the wavepacket* (as distinguished from the energy expended to launch it). Recall in this regard[3] that the least energy investment sufficient to confine a particle to the interior of a one-dimensional box of length ℓ is given by $E_0 = \hbar^2/8m(\ell/2\pi)^2$.

Equations (2) display $\bar{E}$ as a sum of two terms. Mita remarks, in this regard, that the kinetic energy $T = \frac{1}{2}\sum m_i \dot{x}_i^2$ of a *classical <u>system</u>* of particles —if we write $x_i = X + r_i$ and understand X to refer to the center of mass

$$X \equiv \frac{\sum m_i x_i}{\sum m_i} \quad : \quad \text{entails} \quad \sum m_i r_i = 0$$

—can be written in a form

$$T = \frac{1}{2}\left(\sum m_i\right)\dot{X}^2 + \frac{1}{2}\sum m_i \dot{r}_i^2$$
$$= T_{\text{of center of mass}} + T_{\text{relative to center of mass}}$$

typical of the several "splitting theorems" encountered in classical mechanics.[4] It becomes tempting in this light to look upon (2)—though it refers to the physics of a *single* particle—as the expression of a "quantum mechanical splitting theorem."

Our Gaussian wavepackets are, of course, *not eigenfunctions* of **H**. Energy measurements will, therefore, display some inevitable statistical scatter. We look, therefore, to

$$(\Delta E)^2 = \langle \mathbf{H}^2 \rangle - \langle \mathbf{H} \rangle^2$$
$$= \frac{1}{4m^2}\left\{ \langle \mathbf{p}^4 \rangle - \langle \mathbf{p}^2 \rangle^2 \right\}$$

and after *Mathematica*-assisted calculation obtain

$$\begin{aligned} &= \frac{1}{4m^2}\left\{ (m^4v^4 + 6m^2v^2\lambda^2 + 3\lambda^4) - (m^2v^2 + \lambda^2)^2 \right\} \\ &= \lambda^2\left(v^2 + \frac{1}{2m^2}\lambda^2\right) \\ &= 2(\bar{E} + \tfrac{1}{2}mv^2)(\bar{E} - \tfrac{1}{2}mv^2) \\ &= 2(mv^2 + \tfrac{\hbar}{4\tau})\tfrac{\hbar}{4\tau} > 0 \end{aligned} \qquad (3)$$

It is not surprising that ΔE is under the direct control of $\lambda = \Delta p$. I have nothing sharp to say about the specific structure of (3), but am in position to describe a more elegant method for deriving this and similar results:

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[3] See, for example, §**2.2** in D. Griffiths, *Introduction to Quantum Mechanics* (1995),

[4] See CLASSICAL MECHANICS (1983), pages 60–65.

We proceed from the observation that

$$\text{probability that } 0 \leqslant \text{measured energy} \leqslant E$$

$$= \text{probability that } -\sqrt{2mE} \leqslant p \leqslant +\sqrt{2mE}$$

$$= \int_{-\sqrt{2mE}}^{+\sqrt{2mE}} \frac{1}{\lambda\sqrt{2\pi}} \exp\left\{-\tfrac{1}{2}\left[\tfrac{p-mv}{\lambda}\right]^2\right\} dp$$

$$= \int_0^E w(E)\, dE$$

where $w(E)$ describes how we expect the results of energy measurements to be distributed. Immediately

$$w(E) = \frac{d}{dE} \int_{-\sqrt{2mE}}^{+\sqrt{2mE}} \frac{1}{\lambda\sqrt{2\pi}} \exp\left\{-\tfrac{1}{2}\left[\tfrac{p-mv}{\lambda}\right]^2\right\} dp$$

$$= \frac{m}{\lambda\sqrt{2\pi}\sqrt{2mE}}\left[\ \exp\left\{-\left[\tfrac{(2\sqrt{mE}-\sqrt{2}\,mv)^2}{4\lambda^2}\right]\right\}\right. \tag{4}$$

$$\left. + \exp\left\{-\left[\tfrac{(2\sqrt{mE}+\sqrt{2}\,mv)^2}{4\lambda^2}\right]\right\}\right]$$

Mathematica—responding to the command

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Integrate[w(E), {E,0,∞}, Assumptions→{Re[λ]>0, Re[m]>0}]
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—supplies $\int_0^\infty w[E]\, dE = 1$ even though (see the following figure) $w(E)$ is weakly
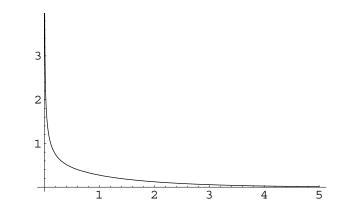

FIGURE 1: *Graph of $w(E)$ in the case $m = v = \lambda = 1$.*

divergent at $E = 0$. Similar commands give back (compare (2) and (3))

$$\bar{E} \equiv \langle E\ \rangle = \int_0^\infty E\ w(e)\, dE = \frac{m^2 v^2 + \lambda^2}{2m}$$

$$\langle E^2 \rangle = \int_0^\infty E^2 w(e)\, dE = \frac{m^4 v^4 + 6m^2 v^2 \lambda^2 + 3\lambda^4}{4m^2}$$

and make it easy to extend the list of expected moments:

$$\langle E^3 \rangle = \int_0^\infty E^3 w(e)\, dE = \frac{m^6 v^6 + 15 m^4 v^4 \lambda^2 + 45 m^2 v^2 \lambda^4 + 15 \lambda^6}{8 m^3}$$

$$\langle E^4 \rangle = \int_0^\infty E^4 w(e)\, dE = \frac{m^8 v^8 + 28 m^6 v^6 \lambda^2 + 210 m^4 v^4 \lambda^4 + 420 m^2 v^2 \lambda^6 + 105 \lambda^8}{16 m^4}$$

$$\vdots$$

"Energy conservation" finds expression in the fact that the moments of all orders (which is to say: $w(E)$ itself) are t-independent.

2. Mita's contribution. Katsunori Mita indicates on his web page[5] that the theory of "probability density and probability density currents in quantum mechanics" is his main research interest. One might therefore anticipate that he prefers to work in the x-representation, and that he entertains a "fluid dynamical" conception of ψ-motion; his recent publications would seem to support that suspicion.[6]

We note by way of preparation that the Gaussian $\psi(x,t)$ of page 2 can be shown by direct (*Mathematica*-assisted) computation to satisfy

$$\left\{ -\tfrac{\hbar^2}{2m} \partial_x^2 - i\hbar \partial_t \right\} \psi(x,t) = 0$$

and

$$\int |\psi(x,t)|^2\, dx = 1 \quad : \quad \text{all } t$$

and that—consistently with results already stated—

$$\langle \mathbf{p} \rangle = -i\hbar \int \psi^*(x,t)\, \partial_x \psi(x,t)\, dx = mv \quad : \quad \text{all } t$$

$$\langle \mathbf{H} \rangle = -(\hbar^2/2m) \int \psi^*(x,t)\, \partial_x^2 \psi(x,t)\, dx = \tfrac{1}{2} m v^2 + \tfrac{1}{4}(\hbar/\tau) \quad : \quad \text{all } t$$

Mita draws attention to the fact that an integration-by-parts supplies

$$\langle \mathbf{H} \rangle = \int \mathcal{E}(x,t)\, dx \tag{5}$$

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[5] www.smcm.edu/nsm/physics/MITA.HTM. We are informed that he chairs the three-person department at St. Mary's College of Maryland, and that he began his career as a string/particle theorist.

[6] "The real part of a wavefunction in tunneling," APJ **62**, 470 (1994); "Virtual probability current associated with the spin," AJP **68**, 259 (2000); "Fluid-like properties of probability densities in quantum mechanics," AJP **69**, 470 (2001).

where

$$\mathcal{E}(x,t) \equiv \tfrac{\hbar^2}{2m}\,\partial_x\psi^*(x,t)\cdot\partial_x\psi(x,t)$$

can (and—when quantum mechanics is developed in the language of classical field theory—definitely would) be looked upon as an "energy density." With Mita[7] we observe that $\mathcal{E}$ can be notated

$$\mathcal{E} = \tfrac{\hbar^2}{2m}\,\frac{|\psi^*\partial_x\psi|^2}{\psi^*\psi} \tag{6}$$

Recall now that

$$\begin{aligned}
\text{probability density} \quad &\rho = \psi^*\psi \\
\text{probability current} \quad &J = i\tfrac{\hbar}{2m}\big(\psi\,\partial_x\psi^* - \psi^*\partial_x\psi\big) \\
&= \tfrac{1}{2m}\big(\psi^*\,\tfrac{\hbar}{i}\partial_x\psi + \text{conjugate}\big) \\
&= \tfrac{1}{m}\mathrm{Re}\big\{\psi^*\,\tfrac{\hbar}{i}\partial_x\psi\big\} \tag{7.1}
\end{aligned}$$

and, drawing inspiration from the last of those equations, define

$$\begin{aligned}
D \equiv &\tfrac{1}{m}\mathrm{Im}\big\{\psi^*\,\tfrac{\hbar}{i}\partial_x\psi\big\} \tag{7.2}\\
= &\tfrac{1}{2mi}\big(\psi^*\,\tfrac{\hbar}{i}\partial_x\psi - \text{conjugate}\big) \\
= &-\tfrac{\hbar}{2m}\big(\psi\,\partial_x\psi^* + \psi^*\partial_x\psi\big) \\
= &-\tfrac{\hbar}{2m}\partial_x\rho
\end{aligned}$$

We then have

$$\tfrac{1}{m}\,\psi^*\,\tfrac{\hbar}{i}\partial_x\psi = J + iD$$

giving

$$\begin{aligned}
\mathcal{E} = &\tfrac{m}{2\rho}\big|\tfrac{1}{m}\,\psi^*\,\tfrac{\hbar}{i}\partial_x\psi\,\big|^2 \\
= &\tfrac{m}{2\rho}(J^2 + D^2) \tag{8}\\
= &\mathcal{E}_{\text{drift}} + \mathcal{E}_{\text{dispersion}} \quad \text{by Mita's interpretation}
\end{aligned}$$

We have here displayed $\mathcal{E}$ as a sum of two (manifestly non-negative) terms, and might claim to have obtained a "quantum mechanical splitting theorem." It is, however, distinct from all previous such theorems, for computation in the Gaussian case gives

$$\int \mathcal{E}_{\text{drift}}(x,t)\,dx = \frac{16m^4v^2\sigma^6 + 4m^2t^2v^2\sigma^2\hbar^2 + t^2\hbar^4}{32m^3\sigma^6 + 8mt^2\sigma^2\hbar^2}$$

which by expansion in powers of $\hbar$ becomes

$$= \tfrac{1}{2}mv^2 + \tfrac{1}{4}(\hbar/\tau)\vartheta^2\Big\{1 - \vartheta^2 + \vartheta^4 - \vartheta^6 + \cdots\Big\}$$

(here $\vartheta \equiv t\hbar/2m\sigma^2 = t/\tau$) and for *all* values of ϑ can be written

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[7] What I present is actually a one-dimensional reorganization of Mita's three-dimensional argument.

$$= \tfrac{1}{2}mv^2 + \tfrac{1}{4}(\hbar/\tau) \cdot \frac{\vartheta^2}{1+\vartheta^2} \tag{9.1}$$

Similarly

$$\int \mathcal{E}_{\text{dispersion}}(x,t)\,dx = \frac{\hbar^2 m\sigma^2}{8m^2\sigma^4 + 2t^2\hbar^2}$$

$$= \tfrac{1}{4}(\hbar/\tau) \cdot \frac{1}{1+\vartheta^2}$$

$$= \tfrac{1}{4}(\hbar/\tau) \cdot \left\{ 1 - \frac{\vartheta^2}{1+\vartheta^2} \right\} \tag{9.2}$$

We note that both

$$E_{\text{drift}}(t) \equiv \int \mathcal{E}_{\text{drift}}(x,t)\,dx \quad \text{and} \quad E_{\text{dispersion}}(t) \equiv \int \mathcal{E}_{\text{dispersion}}(x,t)\,dx$$

are *time-dependent*, though their sum is a familiar constant:

$$E_{\text{drift}}(t) + E_{\text{dispersion}}(t) = \bar{E} \quad : \quad \text{all } t \tag{10}$$

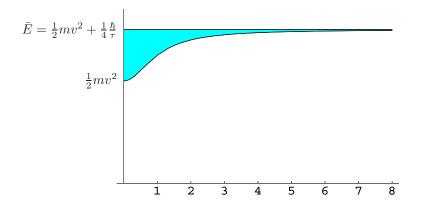The situation is illustrated in the following figure:

FIGURE 2: *The lower curve shows the growth of* $E_{\text{drift}}(t)$, *with time marked off in* τ-*units. The shaded region shows the diminishing contribution made to* $\bar{E}$ *by* $E_{\text{dispersion}}(t)$. *I can think offhand of no way in which such a figure might be* extracted *from observational data:* the point illustrated is, in that sense and to that extent, entirely formal.

Recognizing the physical importance that familiarly attaches to the

$$\text{probability current} \quad J = \tfrac{1}{m}\text{Re}\big\{\psi^* \mathbf{p}\,\psi\big\}$$

Mita takes satisfaction from the fact that he has found work for

$$D = \tfrac{1}{m}\text{Im}\big\{\psi^* \mathbf{p}\,\psi\big\}$$

to do, and devotes the remainder of his paper to an effort to support his claim that D is an "important object, susceptible to useful physical interpretation" [my quotation marks]. I attempt to follow him down this road:

Assuming $\psi(x,t)$ to have the Gaussian form described on page 2, we (with major assistance by *Mathematica*) compute

$$J(x,t) = \frac{v + xt/\tau^2}{1 + (t/\tau)^2} \cdot \underbrace{\frac{1}{\sigma(t)\sqrt{2\pi}} \exp\left\{ -\tfrac{1}{2}\left[\tfrac{x-vt}{\sigma(t)} \right]^2 \right\}}_{\rho(x,t)}$$

$$= v\rho + \frac{t(x - vt)}{\tau^2[1 + (t/\tau)^2]}\rho \tag{11.1}$$

$$D(x,t) = \frac{x - vt}{\tau[1 + (t/\tau)^2]}\rho \tag{11.2}$$

Therefore

$$J = v\rho + (t/\tau)D \tag{12.1}$$

which by (7.2) becomes

$$= v\rho - \tfrac{\hbar t}{2m\tau} \partial_x\rho$$

$$= v\rho - t\tfrac{\hbar^2}{4m^2\sigma^2} \partial_x\rho$$

This Mita would have us read as a statement that

$$= J_{\text{drift}} + J_{\text{dispersion}} \tag{12.2}$$

I would emphasize that (11) and (12) describe (non-obvious) special properties of *launched Gaussian* solutions of the free-particle Schrödinger equation: they do not pertain to free wavepackets-in-general.

From (12.1) it follows that the probability conservation equation

$$\partial_t\rho + \partial_x J = 0$$

can be written

$$(\partial_t + v\partial_x)\rho = t\tfrac{\hbar}{2m\tau} \partial_{xx}\rho \tag{13.1}$$

which <u>in the case $v = 0$</u> assumes the form

$$\downarrow$$
$$\partial_t\rho = t\tfrac{\hbar}{2m\tau} \cdot \partial_{xx}\rho \tag{13.2}$$

of a *diffusion equation with a time-dependent diffusion coefficient.* Equation (13.1) can be recovered from (13.2) by a change of variables

$$\left. \begin{array}{l} t = t \\ x = x - vt \end{array} \right\} \quad \Longrightarrow \quad \left\{ \begin{array}{l} \partial_t = \partial_t + v\partial_x \\ \partial_x = \partial_x \end{array} \right.$$

i.e., by passing to a drifting frame, the assumption here being that $\rho(x,t)$ transforms as a scalar: $\rho(x,t) \longmapsto \rho(x,t) = \rho(x - vt, t)$.

Mita remarks that if we return with (12.1) to (8) we find that the energy density can be described

$$\mathcal{E} = \tfrac{m}{2\rho}\Big[\big(v\rho + (t/\tau)D\big)^2 + D^2\Big]$$
$$= \tfrac{1}{2}mv^2\rho + mv(t/\tau)D + \tfrac{m}{2\rho}\big[1 + (t/\tau)^2\big]D^2$$

Because $D(x,t)$ is—whether one works from (7.2) (*i.e.*, from $D \sim \partial_x \rho$) of from (11.2)—*odd* with respect to the point $x = vt$, it is evident (and confirmed by *Mathematica*) that $\int D\,dx = 0$ (all t) so for purposes of E-evaluation we might as well write

$$\downarrow$$
$$= \tfrac{1}{2}mv^2\rho + \tfrac{m}{2\rho}\big[1 + (t/\tau)^2\big]D^2$$

Trivially $\int \rho\,dx = 1$, while non-trivially $\int \tfrac{m}{2\rho}\big[1 + (t/\tau)^2\big]D^2\,dx = \hbar/4\tau$. So we do in fact recover (1). It becomes tempting at this point—*at least in this specifically Gaussian context*—to adopt these <u>modificiations of the definitions</u> proposed at (9):

$$\mathcal{E}_{\text{drift}} \equiv \tfrac{1}{2}mv^2\rho \tag{14.1}$$
$$\mathcal{E}_{\text{dispersion}} \equiv \tfrac{m}{2\rho}\big[1 + (t/\tau)^2\big]D^2$$
$$= \frac{m(x-vt)^2}{2\tau^2[1 + (t/\tau)^2]}\rho$$
$$= (\hbar/4\tau)\big[\tfrac{x-vt}{\sigma(t)}\big]^2\rho \tag{14.2}$$

Equation (1) can be considered on this basis to follow at once from

$$\int \rho\,dx = \int \big[\tfrac{x-vt}{\sigma(t)}\big]^2\rho\,dx = 1 \quad : \quad \rho = \tfrac{1}{\sigma(t)\sqrt{2\pi}}\exp\Big\{-\tfrac{1}{2}\big[\tfrac{x-vt}{\sigma(t)}\big]^2\Big\}$$

Note especially (in reference to Figure 2) that the E_{drift} and $E_{\text{dispersion}}$ latent in (14) are *time-independent*; we have recovered an "eneregetic splitting theorem" of the type first encountered at the top of page 3.

Mita concludes his Gaussian §4 with the computation-based remark that

$$\int (D^2/\rho)\,dx = \frac{\sigma^2\hbar^2}{4m^2\sigma^4 + t^2\hbar^2}$$
$$= (\hbar/2m\sigma)^2[1 + (t/\tau)^2]^{-1}$$

from which, by $\hbar/2m\sigma = \lambda/m$, he extracts

$$\lambda = \text{constant value of } \Delta p = m\bigg[[1 + (t/\tau)^2]\cdot\int (D^2/\rho)\,dx\bigg]^{\frac{1}{2}}$$

But if one knew the value of τ—as required to evaluate the expression on the right—one would already know the value of $\lambda = \sqrt{2\tau/\hbar m}$, so it is hard to see how the preceding equation can be read as compelling evidence that D is an "important object."

Mita appears to have drawn inspiration from what he considers to be the "interesting graphical representation" of the free motion of a Gaussian wavepacket that appears as Figure 5.3 on page 87 (and also on the cover) of Richard Robinett's *Quantum Mechanics*: *Classical Results, Modern Systems & Visualized Examples* (1997). A slight variant of Robinett's figure is presented as his Figure 1, and my own reconstruction appears below:
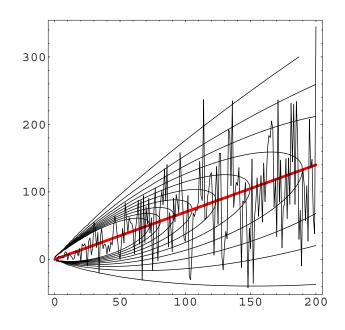

FIGURE 3: *Reconstruction of a figure first constructed by Robinett, and appropriated by Mita. Time t (in τ-units) runs →, and x runs ↑. The contours show the drift and dispersal of the Gaussian*

$$\rho(x,t) = \tfrac{1}{\sigma(t)\sqrt{2\pi}} \, \exp\left\{ -\tfrac{1}{2}\left[\tfrac{x-vt}{\sigma(t)} \right]^2 \right\}$$
$$= \texttt{NormalDistribution[}\ vt, \sigma\sqrt{1+(t/\tau)^2}\ \texttt{]}$$

with $v = 0.7$, $\sigma = 0.5$, $\tau = 1.0$. The bold red line is a graph of the "classical trajectory" $x = vt$. Points were selected randomly from the evolving distribution, connected by lines and plotted by means of the Mathematica commands

```
<<Statistics`Continuous Distributions`
qpath = Table[Random[
        NormalDistribution[.7t,.5√(1+t²)]],{t,1,200,1}]
ListPlot[qpath, PlotJoined→True];
```

Robinett himself appears to attach no particular *physical* significance to his "stochastic meander." But Mita seems inclined to attach *literal* significance to Robinett's figure, to read it as a description of "random oscillations of the particle about its classical trajectory," of "[one of the many] possible random motion[s] of the particle along its classical trajectory." In Mita's view the figure "suggests that the particle executes [meaning that in point of physical fact the particle *does* execute?] analogous classical, random oscillations about $\langle x \rangle$." In the caption to his Figure 1 he allows himself to refer to "*the* random oscillations of the particle" as though such "oscillations" were established realities. Mita's stated objective is to spell out "some interesting implications" of that train of thought. Possibly Mita's words are less seriously intended, more tentative, than I read them. But I find in his paper no phrases of the type "let us pretend;" "it is as if. . . " In this specific regard I admit to some surprise that the AJP referees did not blow their whistles.

The obvious difficulty with Mita's idea, as I read it, is that *if any one of the computed sample points actually reported the <u>result of an observation</u> then "collapse of the wavefunction" would render all subsequent points irrelevant.* If, on the other hand, Mita holds it to be essential that they remain *un*observed, then he has managed simply to encumber ordinary quantum mechanics with an unphysical interpretive overlay. That criticism notwithstanding, Mita has made some interesting points, and he is certainly not the first person to have extracted valuable insight from a misconception.

Mita's reading of Robinett's figure does—obliquely—raise an interesting question: How—in what approximation, and in what limit—does drift survive the observation process? Certainly it is possible in the classical limit to "watch a particle move." In the deep quantum realm we expect repeated observation of a particle in a state described initially by a Gaussian-at-rest to yield data with a time-independent mean. But such a sequence of events, if viewed by an observer in uniform motion, would surely yield a *drifting* mean. We confront, therefore, this question: How, and to what extent, does a "launched Gaussian" manage to retain its motion when (imprecisely?) watched? The problem appears to have much in common with the "quantum Zeno problem," the "cloud chamber problem"[8]. . . and yet to differ from both in important ways. It is a problem—*if* a problem—to which I hope to return on another occasion.

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[8] See pages 383–385 in Griffiths[3] and Chapter 6 in Partha Ghose, *Testing Quantum Mechanics on New Ground* (1999).